

A REMARK ON TWO DUALITY RELATIONS

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ABSTRACT. We remark that an easy combination of two known results yields a positive answer, up to $\log(n)$ terms, to a duality conjecture that goes back to Pietsch. In particular, we show that for any two symmetric convex bodies K, T in \mathbb{R}^n , denoting by $N(K, T)$ the minimal number of translates of T needed to cover K , one has:

$$N(K, T) \leq N(T^\circ, (C \log(n))^{-1} K^\circ)^{C \log(n) \log \log(n)},$$

where K°, T° are the polar bodies to K, T , respectively, and $C \geq 1$ is a universal constant. As a corollary, we observe a new duality result (up to $\log(n)$ terms) for Talagrand's γ_p functionals.

1. INTRODUCTION

Let K and T denote two convex bodies in \mathbb{R}^n (i.e. convex compact sets with non-empty interior). Throughout this note we assume that all bodies in question are centrally symmetric w.r.t. to the origin (e.g. $K = -K$). For a convex body L , we denote by L° its polar body, defined as $L^\circ = \{x \in \mathbb{R}^n; \langle x, y \rangle \leq 1 \ \forall y \in L\}$. The *covering number of K by T* , denoted $N(K, T)$, is defined as the minimal number of translates of T needed to cover K , i.e.:

$$N(K, T) = \min \left\{ N ; \exists x_1, \dots, x_N \in \mathbb{R}^n, K \subset \bigcup_{1 \leq i \leq N} (x_i + T) \right\}.$$

In this note, we address the following conjecture of Pietsch ([Pie72, p. 38]) from 1972, originally formulated in operator-theoretic notations:

Duality Conjecture for Covering Numbers. Do there exist numerical constants $a, b \geq 1$ such that for any dimension n and for any two symmetric convex bodies K, T in \mathbb{R}^n one has:

$$(1.1) \quad b^{-1} \log N(T^\circ, aK^\circ) \leq \log N(K, T) \leq b \log N(T^\circ, a^{-1}K^\circ) ?$$

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This problem may be equivalently formulated using the notion of *entropy numbers*. For a real number $k \geq 0$, denote the k 'th entropy number of K w.r.t. T as:

$$e_k(K, T) = \inf\{\varepsilon > 0; N(K, \varepsilon T) \leq 2^k\}.$$

Then the duality conjecture may be equivalently formulated with (1.1) replaced by:

$$(1.2) \quad a^{-1}e_{bk}(T^\circ, K^\circ) \leq e_k(K, T) \leq ae_{b^{-1}k}(T^\circ, K^\circ)$$

for all $k \geq 0$ (and there is no loss in generality if we assume that k is an integer).

As already mentioned, the duality conjecture originated from operator theory, where entropy numbers are used to quantify the compactness of an operator $u : X \rightarrow Y$ between two Banach spaces. Leaving the finite dimensional setting for a brief moment, if $K = u(B(X))$ and $T = B(Y)$, where $B(Z)$ denotes the unit-ball of a Banach space Z , then it is easy to see that $e_k(K, T) \rightarrow 0$ as $k \rightarrow \infty$ iff the operator u is compact. Since u is compact iff its dual $u^* : Y^* \rightarrow X^*$ is too, and since $u^*(B(Y^*)) = u^*(T^\circ)$ and $B(X^*) = u^*(K^\circ)$, it follows that $e_k(K, T) \rightarrow 0$ iff $e_k(T^\circ, K^\circ) \rightarrow 0$. Hence, it is natural to conjecture that the rate of convergence to 0 is asymptotically similar in both cases. A strong interpretation of this similarity is given by (1.2). We will mention other weaker interpretations below.

Although the general problem is still not completely settled, there has been substantial progress in recent years, and the answer is known to be positive for a wide class of bodies. We begin by describing some results in this direction. We comment here that when the result imposes the same restrictions on K and T , it is obviously enough to specify only one side of the inequalities in (1.1) or (1.2). When both K and T are ellipsoids, it is easy to see that in fact $N(K, T) = N(T^\circ, K^\circ)$. Other special cases were settled in [Sch84], [Car85], [GKS87], [KMTJ90], [PTJ89]. In [KM87], it was shown that:

$$C^{-n}N(T^\circ, K^\circ) \leq N(K, T) \leq C^n N(T^\circ, K^\circ),$$

for some universal constant $C > 1$. This implies that the tail behaviour of the entropy numbers satisfies the duality problem, i.e. $e_{\lambda k}(K, T) \leq 2e_k(T^\circ, K^\circ)$ for some universal constant $\lambda > 0$ and all $k \geq n$. This was subsequently generalized in [Pis89a].

Another variant of the problem, is to consider not the individual entropy numbers, but rather the entire sequences $\{e_k(K, T)\}$ and $\{e_k(T^\circ, K^\circ)\}$. Then one may ask whether:

$$(1.3) \quad C^{-1} \|\{e_k(T^\circ, K^\circ)\}\| \leq \|\{e_k(K, T)\}\| \leq C \|\{e_k(T^\circ, K^\circ)\}\|$$

for some universal constant $C > 1$ and any symmetric (i.e. invariant to permutations) norm $\|\cdot\|$. When one of the bodies is an ellipsoid, this was positively settled in [TJ87]. Later, in [BPSTJ89], this was extended to the case when one of the bodies is uniformly convex or more generally K-convex (see [BPSTJ89] and [Pis89b] for definitions), in which case the constant C in (1.3) depends only on the K-convexity constant. The technique developed in [BPSTJ89] played a crucial role in some of the subsequent results on this problem, and one particular remark will play an essential role in this note.

Returning to the duality problem of individual entropy numbers, it was shown in [MS00] that there exist universal constants $a, b \geq 1$ such that when $T = D$ is an ellipsoid:

$$e_{bk}(D^\circ, K^\circ) \leq a(1 + \log k)^3 e_k(K, D),$$

for all $k \geq 0$. In addition, the authors of [MS00] observed a connection between (one side of) the duality conjecture with $T = D$ and a certain geometric lemma. Later, the case when one of the bodies is an ellipsoid was completely settled in [AMS04], by showing that:

$$b^{-1} \log N(D^\circ, aK^\circ) \leq \log N(K, D) \leq b \log N(D^\circ, a^{-1}K^\circ).$$

The main new tool developed in [AMS04] was the so called “Reduction Lemma”, which roughly reduces the problem (1.1) for all K, T to the case $K \subset 4T$. This will be the second important tool in this note.

Finally, in [AMSTJ04], the Reduction Lemma was combined with the techniques developed in [BPSTJ89], to transfer the results obtained there for the sequence of entropy numbers, to the individual ones. Thus, when one of the bodies K or T is K-convex, (1.1) was shown to hold with the constants a, b depending solely on the K-convexity constant. The key ideological step in [AMSTJ04] was to separate the question of “complexity” from the question of duality, by explicitly introducing a new notion of *convexified packing number*, which was implicitly used in [BPSTJ89]. We will later refer to this new notion as well.

Our first new observation in this note is in fact an immediate consequence of Theorem 6 in [BPSTJ89] and the Reduction Lemma in [AMS04]. It settles the duality problem (1.1) (and (1.2)) up to $\log(n)$ terms, and in fact strengthens and generalizes all previously known results into a single statement. Because of the symmetry between K and T (as explained below), we formulate this as a one sided inequality:

Theorem 1.1. *Let K, T be two symmetric convex bodies in \mathbb{R}^n . Then:*

$$(1.4) \quad \log N(K, T) \leq V \log(V) \log N(T^\circ, V^{-1}K^\circ),$$

where $V = \min(V(K), V(T))$ and $V(L)$ is defined as:

$$(1.5) \quad V(L) := \inf \{ \log(Cd_{BM}(L, B))f(\mathcal{K}(X_B)); B \text{ is a convex body in } \mathbb{R}^n \},$$

where $C > 0$ is a universal constant, and f is a function depending solely on $\mathcal{K}(X_B)$, the K -convexity constant of the Banach space X_B whose unit ball is B .

Recall that the Banach-Mazur distance $d_{BM}(L, B)$ of two symmetric convex bodies L, B is defined as:

$$d_{BM}(L, B) := \inf \{ \gamma \geq 1; B \subset T(L) \subset \gamma B \},$$

where the infimum runs over all linear transformations T . Since $V(L) = V(L^\circ)$ because $d_{BM}(L, B) = d_{BM}(L^\circ, B^\circ)$ and $\mathcal{K}(X_{B^\circ}) = \mathcal{K}(X_B)$, applying the Theorem to $K' = T^\circ$ and $T' = K^\circ$ gives the opposite inequality (with the same V):

$$(1.6) \quad (V \log(V))^{-1} \log N(T^\circ, VK^\circ) \leq \log N(K, T).$$

In addition, since by John's Theorem, the Banach-Mazur distance of any symmetric convex body in \mathbb{R}^n from the Euclidean ball D is at most \sqrt{n} , and since $\mathcal{K}(D) = 1$, we immediately have:

Corollary 1.2. *With the same notations as in Theorem 1.1:*

$$\log N(K, T) \leq C \log(1+n) \log \log(2+n) \log N(T^\circ, (C \log(1+n))^{-1}K^\circ),$$

where $C > 0$ is a universal constant.

This should be compared with the previously known best estimate (to the best of our knowledge) for general symmetric convex bodies K, T :

$$\log N(K, T) \leq C \log N(T^\circ, (Cn)^{-1/2}K^\circ),$$

which is derived by comparing K with its John ellipsoid and using the duality result of [AMS04] for ellipsoids.

Although there has been much progress in recent years towards a positive answer to the duality conjecture, it is still not clear that a positive answer should hold in full generality. In view of the Corollary 1.2, and Pisier's well known estimate $\mathcal{K}(X_B) \leq C \log(1+n)$ for any symmetric convex body B in \mathbb{R}^n , we conjecture a weaker form of the duality problem:

Weak Duality Conjecture for Covering Numbers. Does there exist a numerical constant $C \geq 1$ such that for any dimension n and for any two symmetric convex bodies K, T in \mathbb{R}^n one has:

$$\log N(K, T) \leq V \log N(T^\circ, V^{-1}K^\circ),$$

where $V = C \min(\mathcal{K}(X_K), \mathcal{K}(X_T))$?

We present the proof of Theorem 1.1 and several other connections to previously mentioned notions in Section 2. In Section 3, we give an application of Corollary 1.2 for Talagrand's celebrated γ_p functionals, which was in fact our motivation for seeking a result in the spirit of Corollary 1.2. Recall that for a metric space (M, d) and $p > 0$, $\gamma_p(M, d)$ is defined as:

$$\gamma_p(M, d) := \inf \sup_{x \in M} \sum_{j \geq 0} 2^{j/p} d(x, M_j)$$

where the infimum runs over all *admissible* sets $\{M_j\}$, meaning that $M_j \subset M$ and $|M_j| = 2^{2^j}$ (we refer to [Tal05, Theorem 1.3.5] for the connection to equivalent definitions). For two symmetric convex bodies K, T , let us denote $\gamma_p(K, T) := \gamma_p(K, d_T)$, where d_T is the metric corresponding to the norm induced by T . The $\gamma_2(\cdot, D)$ functional, when D is an ellipsoid, was introduced to study the boundedness of Gaussian processes (see [Tal05] for an historical account on this topic). It was shown by Talagrand in his celebrated "Majorizing Measures Theorem", that in fact $\gamma_2(K, D)$ and $E \sup_{x \in K} \langle x, G \rangle$, where G is a Gaussian r.v. (with covariance corresponding to D in an appropriate manner), are equivalent to within universal constants. This was later extended to various other classes of stochastic processes, where the naturally arising metric d is not the l_2 norm (again we refer to [Tal05] for an account).

Our second observation in this note is the following duality relation for the γ_p functionals:

Theorem 1.3. *Let K, T be two symmetric convex bodies in \mathbb{R}^n . Then for any $p > 0$:*

$$\gamma_p(K, T) \leq C_p \log(1+n)^{2+1/p} \log \log(2+n)^{1/p} \gamma_p(T^\circ, K^\circ),$$

where $C_p > 0$ depends solely on p .

Although we strongly feel that this is unlikely, one could conjecture that the $\log(n)$ terms are not required in the last Theorem (at least for some values of p). In that case, as will be evident from the proof, we mention that such a conjecture is independent of the duality conjecture for covering numbers, in the sense that neither one implies the other.

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2. DUALITY OF ENTROPY

As emphasized in the Introduction, the proof of Theorem 1.1 is immediate once we recall two previously known results. The first is the recently observed “Reduction Lemma” ([AMS04, Proposition 12]), which uses a clever iteration procedure to telescopically expand and reduce the appearing terms. We carefully formulate it below:

Theorem 2.1 ([AMS04]). *Let T be a convex symmetric body in a Euclidean space such that, for some constants $a, b \geq 1$, for any convex symmetric body $K \subset 4T$, one has:*

$$\log N(K, T) \leq b \log N(T^\circ, a^{-1}K^\circ).$$

Then for any convex symmetric body K :

$$\log N(K, T) \leq b \log_2(48a) \log N(T^\circ, (8a)^{-1}K^\circ).$$

Dually, if K is fixed and the hypothesis holds for all T verifying $K \subset 4T$, then the conclusion holds for any T .

The second known result goes back to the work of [BPSTJ89]. It uses the so called Maurey’s Lemma, which (roughly speaking) estimates the covering number of the convex hull of m points by the unit-ball of a K -convex space. We combine Theorem 6 and the subsequent remark from [BPSTJ89] into the following:

Theorem 2.2 ([BPSTJ89]). *Let K, T be two symmetric convex bodies in \mathbb{R}^n , such that $K \subset 4T$. Then:*

$$\log N(K, T) \leq V \log N(T^\circ, V^{-1}K^\circ),$$

where $V = \min(V(K), V(T))$ and $V(L)$ is given by (1.5).

Combining these two results, we immediately deduce Theorem 1.1. Note that if $V = V(T)$ in Theorem 1.1, we proceed by fixing T , applying Theorem 2.2 for all K satisfying $K \subset 4T$ and use the first part of the Reduction Lemma to deduce (1.4) for all K ; if $V = V(K)$, we fix K and repeat the argument by interchanging the roles of K and T and using the second part of the Reduction Lemma.

We remark that the proof of Theorem 2.2 in fact gives an explicit expression for V , rather than the implicit one used in (1.5):

$$(2.1) \quad V := C_1 \inf \{ \log(C_2 \gamma) (10T_p(X_B))^q; K \subset \gamma B, B \subset 4T, \gamma \geq 1 \},$$

where the infimum runs over all symmetric convex bodies B in \mathbb{R}^n , $T_p(X_B)$ is the type p ($1 < p \leq 2$) constant of the Banach space X_B whose unit-ball is B , $q = p^* = p/(p-1)$ and $C_1, C_2 \geq 1$ are two universal constants (see [MS86] for the definition of type). Theorem 1.1 was formulated using an implicit function f of $\mathcal{K}(X_B)$, since by several important results of Pisier ([Pis73],[Pis82]), an infinite dimensional Banach Space is K -convex iff it has some non-trivial type $p > 1$. We comment that in [Pis82], an explicit formula bounding $\mathcal{K}(X_B)$ as a function of $T_p(X_B)$ and q was obtained. It is possible to obtain an explicit reverse bound using the results in [Pis83], but it is much easier to use an abstract argument which infers the existence of a $p > 1$, depending solely on $\mathcal{K}(X_B)$, such that $T_p(X_B)$ depends solely on $\mathcal{K}(X_B)$ (see, e.g. [KM06, Lemma 4.2]). The advantage of using the K -convexity constant $\mathcal{K}(X_B)$ (instead of $T_p(X_B)$ and q), lies in the fact that we may use duality and deduce the other side of the duality inequality (1.6) with the same V , as explained in the Introduction. We also remark that once V in (2.1) is expressed using $\mathcal{K}(X_B)$, it is clear that $V \leq \min(V(K), V(T))$ where $V(L)$ is given by (1.5). We need this “separable” estimate on V , so that we may apply the Reduction Lemma (where the estimate on one of the bodies must be fixed).

It is important to note that the proof of Theorem 2.2 actually connects the notions of covering and convexified packing, mentioned in the Introduction. For two symmetric convex bodies K and T , the convexified packing number, or convex separation number, was defined in [AMSTJ04] as:

$$\hat{M}(K, T) = \max \left\{ N; \begin{array}{l} \exists x_1, \dots, x_N \in K \text{ such that} \\ (x_j + \text{int}T) \cap \text{conv} \{x_i; i < j\} = \emptyset \end{array} \right\}.$$

Here $\text{int}(T)$ denotes the interior of the set T . Note that we always have $\hat{M}(K, T) \leq N(K, T/2)$ by a standard argument (see [AMSTJ04]). Then the proof actually shows:

Theorem 2.3 ([BPSTJ89]). *Under the same conditions as in Theorem 2.2:*

$$\log N(K, T) \leq V \log \hat{M}(K, V^{-1}T).$$

Using John’s Theorem as in Corollary 1.2, we have:

Corollary 2.4 ([BPSTJ89]). *Let K, T be two symmetric convex bodies in \mathbb{R}^n , such that $K \subset 4T$. Then:*

$$\log N(K, T) \leq C \log(1+n) \log \hat{M}(K, (C \log(1+n))^{-1}T).$$

We mention these variants of Theorem 1.1 and Corollary 1.2 here, because the framework developed in [AMSTJ04] suggests that this is the correct way to understand the duality problem. The cost of transition from covering to convex separation, as given by Theorem 2.3 and Corollary 2.4, is a certain measure of the complexity of the bodies involved. Once the transition is achieved, the duality framework developed in [AMS04] and [AMSTJ04] finishes the job. Indeed, it was shown in [AMSTJ04] that the convex separation numbers always satisfy a duality relation, for any pair of symmetric convex bodies K, T :

$$\hat{M}(K, T) \leq \hat{M}(T^\circ, K^\circ/2)^2.$$

Using Theorem 2.3, we conclude that when $K \subset 4T$:

$$\begin{aligned} \log N(K, T) &\leq V \log \hat{M}(K, V^{-1}T) \leq 2V \log \hat{M}(T^\circ, (2V)^{-1}K^\circ) \\ &\leq 2V \log N(T^\circ, (4V)^{-1}K^\circ). \end{aligned}$$

The Reduction Lemma now immediately gives Theorem 1.1.

To conclude this section, we mention that Theorem 2.3 is already stronger than all of the results in [AMSTJ04] connecting the covering and the convex separation numbers. The technique involving the use of Maurey's Lemma, which was also used in [AMSTJ04] (see also [Art04]), is optimally exploited in the proof of Theorem 2.3 (Theorem 6 in [BPSTJ89]), by using a clever iteration procedure, producing the log factor in the various definitions (1.5) and (2.1) of V . All previous general results (with no restriction on K and T) pay a linear penalty in the Banach-Mazur distance from "low-complexity" bodies, which may be as large as \sqrt{n} .

3. DUALITY OF TALAGRAND'S γ_p FUNCTIONALS

Given Corollary 1.2, proving Theorem 1.3 is rather elementary, although it involves an analogue to Sudakov's Minoration bound which we have not encountered elsewhere. Before proceeding, we remark that for our purposes, it is totally immaterial whether the points $\{x_i\}$ in the definition of $N(K, T)$ are chosen to lie inside K or not. Indeed, denoting by $N'(K, T)$ the variant where the points *are* required to lie inside K , it is elementary to check that:

$$N'(K, 2T) \leq N(K, T) \leq N'(K, T).$$

Since throughout this note we allow the insertion of homothety constants in all expressions, or multiplying the entropy numbers by universal constants, this lack of distinction is well justified.

First, recall that by Dudley's entropy bound ([Tal05]):

$$(3.1) \quad \gamma_p(K, T) \leq C_p \sum_{k \geq 1} k^{1/p-1} e_k(K, T),$$

where $C_p > 0$ is some constant depending on p . The argument is elementary:

$$\gamma_p(K, T) := \inf \sup_{x \in K} \sum_{j \geq 0} 2^{j/p} d_T(x, K_j) \leq \inf \sum_{j \geq 0} 2^{j/p} \sup_{x \in K} d_T(x, K_j).$$

Choosing K_j to be the set of 2^{2^j} points (inside K) attaining the minimum in the definition of $N(K, e_{2^j}(K, T)T)$, we see that:

$$\gamma_p(K, T) \leq \sum_{j \geq 0} 2^{j/p} e_{2^j}(K, T).$$

It is elementary to verify that for $p \geq 1$ and $j \geq 1$:

$$2^{j/p} \leq C_p \sum_{k=2^{j-1}}^{2^j-1} k^{1/p-1},$$

where $C_p = (p(1 - 2^{-1/p}))^{-1}$. Since e_k is a non-increasing sequence, we have:

$$\begin{aligned} \gamma_p(K, T) &\leq e_1(K, T) + C_p \sum_{j \geq 1} \sum_{k=2^{j-1}}^{2^j-1} k^{1/p-1} e_{2^j}(K, T) \\ &\leq e_1(K, T) + C_p \sum_{k \geq 1} k^{1/p-1} e_k(K, T) \leq (1 + C_p) \sum_{k \geq 1} k^{1/p-1} e_k(K, T). \end{aligned}$$

A similar argument works for $0 < p < 1$.

Dudley's entropy upper bound appears naturally when studying the supremum of Gaussian processes on a set K , e.g. $E \sup_{x \in K} \langle x, G \rangle$ where G is a Gaussian r.v. As mentioned in the Introduction, a deep theorem of Talagrand asserts that the latter expectancy is in fact equivalent (to within universal constants) to $\gamma_2(K, D)$ where D is an ellipsoid corresponding to the covariance of G . The corresponding lower bound on $E \sup_{x \in K} \langle x, G \rangle$ is due to Sudakov ([Sud71]):

$$E \sup_{x \in K} \langle x, G \rangle \geq c \sup_{k \geq 1} k^{1/2} e_k(K, D).$$

When the body T is not an ellipsoid or when $p \neq 2$, there is no direct connection between Gaussian processes and $\gamma_p(K, T)$. Nevertheless, we note that the analogue to Sudakov's Minoration bound holds in full generality:

Lemma 3.1.

$$\gamma_p(K, T) \geq 2^{-1/p} \sup_{k \geq 1} k^{1/p} e_k(K, T).$$

Proof. Let $k \geq 1$ be given, and let $j \geq 0$ be such that $2^j \leq k < 2^{j+1}$. Then:

$$\gamma_p(K, T) := \inf \sup_{x \in K} \sum_{l \geq 0} 2^{l/p} d_T(x, K_l) \geq \inf \sup_{x \in K} 2^{j/p} d_T(x, K_j).$$

Since for any admissible set K_j we have $|K_j| = 2^{2^j} \leq 2^k$, it follows by definition that $\sup_{x \in K} d_T(x, K_j) \geq e_k(K, T)$. Hence:

$$\gamma_p(K, T) \geq 2^{j/p} e_k(K, T) \geq 2^{-1/p} k^{1/p} e_k(K, T).$$

Since $k \geq 1$ was arbitrary, the assertion follows. \square

We will need one last lemma for the proof of Theorem 1.3:

Lemma 3.2. *For all $k \geq 3n$:*

$$e_k(K, T) \leq 2e_n(K, T) \exp(-ck/n),$$

where $c > 0$ is some universal constant.

Proof. Denote $e_k = e_k(K, T)$ and $e_n = e_n(K, T)$ for short. W.l.o.g. we assume that $N(K, e_k T) = 2^k$ and $N(K, e_n T) = 2^n$. Obviously we have:

$$(3.2) \quad N(K, e_k T) \leq N(K, e_n T) N(e_n T, e_k T).$$

Also $N(e_n T, e_k T) = N(T, \frac{e_k}{e_n} T)$, and by a standard volume estimation argument, we can find an e_k/e_n T -net of T with cardinality no greater than $(1 + \frac{2e_n}{e_k})^n$. Plugging everything into (3.2), we see that:

$$2^k \leq 2^n \left(1 + \frac{2e_n}{e_k}\right)^n,$$

or equivalently:

$$\exp\left(\log(2) \frac{k-n}{n}\right) - 1 \leq \frac{2e_n}{e_k}.$$

Since $k \geq 3n$, we can find a universal constant $c > 0$ such that:

$$\exp\left(\log(2) \frac{k-n}{n}\right) - 1 \geq \exp\left(c \frac{k}{n}\right).$$

The assertion now readily follows. \square

We can now deduce the following equivalence, up to a $\log(n)$ term, of the γ_p functional, Sudakov's lower bound and Dudley's upper bound. Although this is probably known, we did not find a reference for it, so we include a proof for completeness.

Proposition 3.3. *Let K, T denote two symmetric convex bodies in \mathbb{R}^n , and denote $e_k = e_k(K, T)$ and $\gamma_p = \gamma_p(K, T)$ for short. Then for any $p > 0$:*

$$2^{-1/p} \sup_{k \geq 1} k^{1/p} e_k \leq \gamma_p \leq C_p \sum_{k \geq 1} k^{1/p-1} e_k \leq \log(1+n) C'_p \sup_{k \geq 1} k^{1/p} e_k,$$

where $C_p, C'_p > 0$ are universal constants depending solely on p .

Proof. The first inequality is Sudakov's lower bound (Lemma 3.1) and the second one is Dudley's upper bound (3.1). We will show the third inequality. Let us split the sum $\sum_{k \geq 1} k^{1/p-1} e_k$ into two parts, up to and from $k = 3n$. For the first part, we obviously have:

$$\sum_{k=1}^{3n-1} k^{1/p-1} e_k \leq \sum_{k=1}^{3n-1} \frac{1}{k} \sup_{k \geq 1} k^{1/p} e_k \leq C \log(1+n) \sup_{k \geq 1} k^{1/p} e_k.$$

We use Lemma 3.2 to evaluate the second sum:

$$\sum_{k \geq 3n} k^{1/p-1} e_k \leq 2e_n \sum_{k \geq 3n} k^{1/p-1} \exp(-ck/n).$$

For $p \geq 1$, $k^{1/p-1}$ is non-increasing, so we use:

$$\begin{aligned} \sum_{k \geq 3n} k^{1/p-1} \exp(-ck/n) &\leq (3n)^{1/p-1} \sum_{k \geq 3n} \exp(-ck/n) \\ &= (3n)^{1/p-1} \frac{\exp(-3c)}{1 - \exp(-c/n)} \leq (3n)^{1/p-1} \exp(-3c) \frac{n}{c} \leq C n^{1/p}. \end{aligned}$$

For $0 < p < 1$, we evaluate the sum with an integral (although the series may not be monotone, it has at most one extremal point, and this can be handled by a loose estimate):

$$\begin{aligned} \sum_{k \geq 3n} k^{1/p-1} \exp(-ck/n) &\leq 3 \int_{3n-1}^{\infty} x^{1/p-1} \exp(-cx/n) dx \\ &\leq 3 \left(\frac{n}{c}\right)^{1/p} \int_0^{\infty} x^{1/p-1} \exp(-x) dx = 3c^{-1/p} \Gamma(1/p) n^{1/p}. \end{aligned}$$

We conclude that in both cases:

$$\sum_{k \geq 3n} k^{1/p-1} e_k \leq C'_p n^{1/p} e_n \leq C'_p \sup_{k \geq 1} k^{1/p} e_k.$$

Summing the two parts together, we conclude the proof. \square

Using Corollary 1.2, the proof of Theorem 1.3 is now clear:

Proof of Theorem 1.3. Corollary 1.2 implies that:

$$e_k(K, T) \leq C \log(1+n) e_{k/(C \log(1+n) \log \log(2+n))}(T^\circ, K^\circ),$$

for some universal constant $C \geq 1$ and all $k \geq 0$. Using Proposition 3.3 twice, we conclude:

$$\begin{aligned} \gamma_p(K, T) &\leq C'_p \log(1+n) \sup_{k \geq 1} k^{1/p} e_k(K, T) \\ &\leq C''_p \log(1+n)^2 (\log(1+n) \log \log(2+n))^{1/p} \sup_{k \geq 1} k^{1/p} e_k(T^\circ, K^\circ) \\ &\leq C_p \log(1+n)^{2+1/p} \log \log(2+n)^{1/p} \gamma_p(T^\circ, K^\circ) \end{aligned}$$

□

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